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Quantum Logic of Sequential Events and Their Objectivistic Probabilities

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A propositional calculus for quantum mechanical systems is presented which formalizes "sequential" connectives "and then" and "or then" for yes–no experiments in the framework of complex Hilbert space. Properties of these connectives are compared with some well-known lattice-theoretical results in quantum logic. Probabilities and objectivization versus the Copenhagen interpretation are discussed in connection with Young's two-slit experiment.

1. INTRODUCTION

The lattice-theoretical results obtained by Jauch (1968) and Piron (1964, 1976) and the quantum-logical approach based on dialog semantics developed by Mittelstaedt (1978) have shown that the lattice L_q of subspaces of a complex Hilbert space H is a good candidate to model the propositional logic of a quantum-mechanical system: subspaces and their projections $P, Q, \ldots$ represent measurable properties of the system and the usual lattice-theoretical constants $\wedge$ (infimum) and $\vee$ (supremum), which in H have the meaning of intersection and join, are interpreted as the connectives "and" and "or"; and the relation $\leqslant$, represented in H by inclusion, takes the role of logical implication between propositions. Since L_q is not a distributive lattice, and so a fortiori not Boolean, certain formulas from classical logic are no longer valid. An important implication no longer generally true is

$$P \leqslant (Q \wedge P) + (Q^{\perp} \wedge P)$$

This special case of distributivity is also closely related to the possibility (or failure, see below, Section 5) to objectivize quantum-mechanical properties, e.g., in the sense that a system possesses, at least with a certain probability, a property P independent of any measurement of P.

In this way, quantum logic is closely connected with a suitable probability theory, and both have to stand up to the problem of objectivity versus the most popular nonobjectivistic alternative: the Copenhagen interpretation.

Since 1959, Mittelstaedt (1959) has outlined and developed in detail a quantum logic (QL) which is complete and consistent with respect to Lorenzen's dialog semantics (Stachow, 1976), and which fits together well with a widely accepted notion of quantum probability theory (Jauch, 1968). Furthermore, his restrictive calculus Q_{eff} of effective QL permits an objectivistic interpretation of quantum-theoretical propositions and their probabilities.

The present paper is written in this spirit. We propose, however, a propositional calculus different from Q_{eff}: the algebra of sequential events, modeled by projections in H, but with new connectives $\sqcap$ ("and then") and $\sqcup$ ("or then"). These are derived from the spectral resolution of the (unnormalized) conditional probability operator PQP (for pure state) in the following way: $P \sqcap Q$ is the projection onto the range (support) of PQP, a definition reminiscent of J. von Neumann's famous projection rule (1955, pp. 200–201). Then $\sqcup$ is introduced by

$$P \sqcup Q := (P^{\perp} \sqcap Q^{\perp})^{\perp}$$

$^{\perp}$ denoting the orthocomplement in H, and a "material implication" $\rightarrow$ is defined by

$$P \rightarrow Q := P^{\perp} \sqcup Q$$

$P \sqcap Q$, $P \sqcup Q$, and $P \rightarrow Q$ are projections, i.e. possible properties of a system, not just observables like PQP. It turns out that $\rightarrow$ is exactly Mittelstaedt's "material quasi-implication" (1970, 1972; see also Hardegree, 1976, and further references quoted there).

A system of projections closed under $\sqcap$, $\sqcup$, $\rightarrow$, $^{\perp}$ will be called a sequential algebra (of propositions, yes–no statements, or projections). This algebra is not a lattice with respect to $\leqslant$, $\sqcap$, $\sqcup$. Nevertheless, it shares all the properties of Q_{eff} *except* for two rules which will be shown to be equivalent to general commensurability and thus to classical logic. Therefore, it is imperative to sacrifice them.

But we shall gain something, too. The above-mentioned special case of (sub-) distributivity becomes true again:

$$P \leqslant Q \sqcap P + Q^{\perp} \sqcap P$$

This inclusion for all P and Q permits the objectivization which led to a contradiction in Boolean logic: in the example of Young's two-slit-experiment we may again say (in our "sequential logic") that if a photon hits the screen behind the two slits, it has passed before through exactly one of the two (which one was actually chosen is known with a certain probability only).

This may sound provocative for Copenhagenians; it does not, however, contradict the well-known phenomenon of interference. This point is discussed in Section 5.

It is hoped that the approach sketched in this paper is well adapted to the intrinsic nature of quantum events, even though it digresses one step further from classical propositional logic and even from a calculus as successful as Mittelstaedt's Q_{eff}.

2. SEQUENTIAL EVENTS

Let P and Q be two projections in a complex Hilbert space H. Because of the one-to-one correspondence between projections and their ranges, we denote the range of P by P as well, so that

$$Px = x$$

and

$$x \in P$$

have the same meaning. If P and Q do not commute, PQ and QP are not projections any more, not even "observables," so that in general no physical meaning can be given to these products. In order to interpret the intersection $P \wedge Q$ and the join $P \vee Q$ as propositions derived from P and Q, we go back to von Neumann's original projection rule, and apply this to the conditional probability operator PQP. Let $E_0(PQP)$ denote the null space of PQP:

$$E_0(PQP) = \{x \in H \mid PQPx = 0\}$$

2.1. *Definition*. The orthocomplement of $E_0(PQP)$, the support or range of PQP, will be written as $P \sqcap Q$:

$$E_0^{\perp}(PQP) = R(PQP) = P \sqcap Q$$

read: "P and then Q."

It follows immediately from $\langle x, PQPx \rangle = \| PQx \|^2$, that

$$PQPx = 0 \quad \text{iff} \quad QPx = 0$$

Now the following representation of $E_0(PQP) = E_0(QP)$ is easily established:

2.2. *Theorem*. For all projections P, Q in H

$$E_0(PQP) = P^{\perp} \vee (P \wedge Q^{\perp})$$

where $\vee$ may be equivalently replaced by $+$.

Proof. See Hardegree (1976), p. 62.

2.3. *Corollary*.

$$P \sqcap Q = P \wedge (P^{\perp} \vee Q).$$

Corollary 2.3 gives a justification for our reading of $P \sqcap Q$ as "P and then Q"; for if we interpret the right-hand side in (2.3) classically, we see that $P \sqcap Q$ is true iff P is true and it is true that Q follows "materially" form P. It is then clear how to define $P \sqcup Q$ "P or then Q," and a "quasi-implication" or "conditional implication" $P \rightarrow Q$:

2.4. *Definitions*. For all projections P, Q in H put

$$\begin{aligned} P \sqcup Q &:= (P^{\perp} \sqcap Q^{\perp})^{\perp} \\ &= E_0(P^{\perp} Q^{\perp} P^{\perp}) = \{x \mid Q^{\perp} P^{\perp} x = 0\} \\ &= P \vee (P^{\perp} \wedge Q) \\ P \rightarrow Q &:= P^{\perp} \sqcup Q \\ &= E_0(P Q^{\perp} P) = \{x \mid Q^{\perp} P x = 0\} \\ &= P^{\perp} \vee (P \wedge Q) \end{aligned}$$

"$P \rightarrow Q$" can be read as follows: "$P \rightarrow Q$ is true iff either $P^{\perp}$ is true or the occurrence of the yes outcome of P leaves the system in a state that makes true Q. In other words, it is true that P quasi-implies Q iff either $P^{\perp}$ is true or the conditional probability of Q given P is equal to 1." (cf. Beltrametti and Cassinelli, 1977, p. 378)—hence the name "material conditional," e.g., in Hardegree (1976). We shall, in the sequel, add the adjective "sequential" whenever we mean $\sqcap$, $\sqcup$, or $\to$, in contrast to the "classical" $\wedge$, $\vee$, or material implication.

2.5. *Definition*. A family of projections in H which is closed under finite applications of $\sqcap$, $\sqcup$, $\to$, and orthocomplementation is called a sequential algebra (of events, properties, propositions).

A sequential algebra is closed also with respect to the logical connectives $\wedge$ and $\vee$:

2.6. *Lemma*. For projections P, Q in H

$$P\wedge Q = P\sqcap(P^{\perp}\sqcup Q) = Q\sqcap(Q^{\perp}\sqcup P)$$

$$P\vee Q = P\sqcup(P^{\perp}\sqcap Q) = Q\sqcup(Q^{\perp}\sqcap P)$$

Proof. From $P\leqslant P\vee Q$ it follows that $P\vee Q - P = (P\vee Q)\wedge P^{\perp} = P^{\perp}\sqcap Q$. The rest is obvious. ∎

These identities display a curious dual symmetry to the formulas in Corollary 2.3 and Definitions 2.4. Moreover, we may interpret the meet of two projections, even if they do not commute, as: $P\wedge Q$ holds if and only if: first P, and then Q follows (materially) from P.

A corresponding operational understanding of sequential connectives in a more general framework of dialog semantics has been given in a forthcoming article by E. -W. Stachow. In this way, controversies in connection with the interpretation of the subspaces $P\wedge Q$ can be avoided (cf. the concise summary in Jammer, 1974, pp. 353–361). The meet and join can also be expressed in terms of the spectral measure of the observable PQP:

2.7. *Theorem*. For all projections P, Q in H,

$$P\wedge Q = E_1(PQP) = E_1(QPQ)$$

where E_1 is the respective projection onto the eigenspace with the eigenvalue 1.

Proof. $x\in E_1(PQP)$ iff $PQPx = x$. From $\|x\|^2 = \langle PQPx, x\rangle = \|QPx\|^2 \leqslant \|Px\|^2 \leqslant \|x\|^2$ we see that $\|Px\| = \|x\|$, i.e., $x\in P$, and also $QPx = x$; and thus together with $Px = x$ that $Qx = x$. The converse is evident. ∎

2.8. *Corollary.*

$$P \vee Q = E_1^{\perp}(P^{\perp}Q^{\perp}P^{\perp})$$

$$P \vee Q \geqslant P \sqcup Q$$

$$P \wedge Q \leqslant P \sqcap Q$$

Before entering a deeper study of sequential events, let us briefly discuss some differences between the *observable* QPQ and the *property* $Q \sqcap P$.

Prima facie, both operators are connected with the notion of a succession or sequential order: Q comes somehow prior to P in the observation of a physical system. More detailed, but still only informally discussed, we may perhaps fix ideas and get a feeling about the difference as follows (assuming discrete spectra): the equality $QPQ\psi = \lambda\psi$, or $\langle Q\psi, PQ\psi \rangle = \lambda\|\psi\|^2, \lambda > 0$, may be read like this: ψ is an eigenstate (objective property) of the observable QPQ and in this state ψ, the attribute: "the system has property P after property Q has been measured" yields or is observed at a value $\lambda > 0$.

It follows $\psi \in R(QPQ)$. For this ψ, $Q \sqcap P\psi = \psi$, i.e., "the system is such that property P after property Q will show," or "ψ is objective with respect to QPQ and with a positive λ" whose exact value does not matter. So $Q \sqcap P$ *decides* [von Neumann's term, see von Neumann (1955), p. 254] that QPQ lies somewhere in $(0, 1]$, whereas QPQ gives the special realization or measurement. Hence, QPQ contains more specific information on the system; $Q \sqcap P$ only indicates if a positive measurement on QPQ is made or not. $Q \sqcap P$ is a possible compound property of the system to be decided in a yes–no experiment. QPQ, in contrast, can express more fine-structure information of the system by way of its spectral decomposition.

The observables $Q \sqcap P$ and QPQ are commensurable:

$$Q \sqcap P \cdot QPQ = QPQ \cdot Q \sqcap P = QPQ$$

which means: a joint measurement, although simultaneously possible, does not yield more information than QPQ can already give.

We shall take up this discussion again at the end of this paper, in Section 5, when probabilities enter the description of quantum mechanical systems.

3. THE LOGIC OF SEQUENTIAL EVENTS

It is well known that for commuting projections, the above connectives $\sqcap$, $\sqcup$, $\rightarrow$ reduce to the classical $\wedge$, $\vee$, and material implication. Let us give a sample of criteria for commutativity of two projections P, Q in H.

3.1. *Theorem.* The following statements are equivalent:

(1) $PQ=QP$
(2) $P\to(Q\to P)=H$
(3) $P\leqslant(Q\to P)$
(4) $P\wedge Q=P\sqcap Q$
(5) $P\vee Q=P\sqcup Q$
(6) $P\sqcap Q=Q\sqcap P$
(7) $P\sqcup Q=Q\sqcup P$
(8) $PQP=QPQ$
(9) $Q\leqslant P\sqcup Q$
(10) $Q\geqslant P\sqcap Q$

Proof. The equivalence of (1)–(5) has been proved by several authors (see, e.g., Piron, 1964, 1976.) (1)$\Leftrightarrow$(6): the direction $\Rightarrow$ is clear; for the reverse, using (4), we have to show that $P\sqcap Q=Q\sqcap P$ implies $P\sqcap Q=P\wedge Q$. We know $P\sqcap Q\geqslant P\wedge Q$. On the other hand, if $x\in P\sqcap Q=Q\sqcap P$, also $x\in P$ and $x\in Q$ (from 2.3!), i.e., $x\in P\wedge Q$. (6) and (7) are equivalent, because $P^{\perp}$ and $Q^{\perp}$ commute iff P and Q commute. To prove (8), we observe that $PQ=QP$ iff $P\wedge Q=P\sqcap Q$ (4), i.e., $E_1(PQP)=E_0^{\perp}(PQP)$. This latter equality is true iff PQP is a projection. But, using (8) twice, we get $(PQP)^2=PQP\cdot PQP=PQ(PQP)=PQ(QPQ)=P(QPQ)=P(PQP)=PQP$, so that PQP is in fact a projection. (9) follows readily from (5) and

$$P\leqslant P\sqcup Q=E_0(P^{\perp}Q^{\perp}P^{\perp})\leqslant E_1^{\perp}(P^{\perp}Q^{\perp}P^{\perp})=P\vee Q$$

similarly for (10). ■

From the criteria we see that our connectives, if they are to be different from the classical ones, can neither commute nor are $P\sqcap Q$ and $P\sqcup Q$ the infimum and supremum of P and Q with respect to inclusion. As a consequence, sequential algebras do not form a lattice with respect to $\sqcap$, $\sqcup$ and $\leqslant$. Furthermore, we do not have associativity of $\sqcap$ or $\sqcup$, $\to$ is not transitive, and the rule of contraposition

$$P\to Q=Q^{\perp}\to P^{\perp}$$

is not valid (cf. Hardegree, 1976, p. 64). In the following, we want to find rules that can serve as generalizations and substitutes for commutativity and contraposition, and then investigate associativity. The next result is crucial:

3.2. *Theorem.* For all projections P,Q in H

(1) $P\cdot Q\sqcap P=P\sqcap Q\cdot Q$, and by taking adjoints:
(2) $Q\sqcap P\cdot P=Q\cdot P\sqcap Q$.

Proof of (1). For $y \in H$ write $y = y_1 + y_2$, with $y_1 \in Q \sqcap P$ and $y_2 \in E_0(QPQ)$. Then $y_1 \in Q$, i.e., $y_1 = Qy_1$, and $Qy_2 \in P^\perp \leqslant (P \sqcap Q)^\perp$. Hence $P \cdot Q \sqcap P y_1 = P y_1 = PQy_1$. On the other hand, $P \sqcap Q \cdot Qy = P \sqcap Q \cdot P y_1 + P \sqcap Q \cdot Qy_2$, where the second term is 0 because of $Qy_2 \in (P \sqcap Q)^\perp$. As $P \geqslant P \sqcap Q$ holds, we may write $P \sqcap Q \cdot Qy_1 = P \sqcap Q \cdot PQy_1$, which equals PQy_1 if we can show $PQy_1 \in P \sqcap Q$ or $\langle PQy_1, x \rangle = 0$ for all x with $QPx = 0$; but $\langle PQy_1, x \rangle = \langle y_1, QPx \rangle = 0$! A more elegant and sophisticated proof of (3.2) (1) follows from

$$\langle P \cdot QPQx, y \rangle = \int_{(0,1]} \lambda d \langle P \cdot E_\lambda (QPQ) x, y \rangle$$

$$\langle PQP \cdot Qx, y \rangle = \int_{(0,1]} \lambda d \langle E_\lambda (PQP) \cdot Qx, y \rangle$$

since the left-hand terms are identical and the non-zero specta of QPQ and PQP coincide, the spectral measures must also be identical, and in particular (1) is valid. ■

With (3.2) we arrive immediately at two laws of contraposition:

3.3. *Corollary.*
(3) $Q \sqcap P \to P^\perp = Q \to (P \sqcap Q)^\perp$ or, which is the same:
(3′) $(Q \sqcap P)^\perp \sqcup P^\perp = Q^\perp \sqcup (P \sqcap Q)^\perp$
(4) $P \to (Q \sqcap P)^\perp = P \sqcap Q \to Q^\perp$, or which is the same:
(4′) $P^\perp \sqcup (Q \sqcap P)^\perp = (P \sqcap Q)^\perp \sqcup Q^\perp$. ■

Proof. From the definition of $\to$; see (2.4).
(3′) and (4′) may be written as

$$(3'') \quad (Q \sqcap P) \sqcap P = Q \sqcap (P \sqcap Q)$$

$$(4'') \quad P \sqcap (Q \sqcap P) = (P \sqcap Q) \sqcap Q$$

these two equalities mix commutativity with associativity and are valid without further assumptions. Observe, however, that $Q \sqcap (Q \sqcap P) = Q \sqcap P, P \sqcap (P \sqcap Q) = P \sqcap Q$, so that $Q \sqcap (Q \sqcap P) = P \sqcap (P \sqcap Q)$ iff P and Q commute! The relationship of $Q \sqcap P$ to PQP and of $P \sqcap Q$ to QPQ is given in the following corollary.

3.4. *Corollary.*
(5) $PQP = P \cdot Q \sqcap P \cdot P$
(6) $QPQ = Q \cdot P \sqcap Q \cdot Q$

Proof.

$$\langle PQPx,y\rangle = \langle P\sqcap Q\cdot PQPx,y\rangle = \langle P\cdot P\sqcap Q\cdot QPx,y\rangle$$

$$\langle P\cdot P\cdot Q\sqcap P\cdot Px,y\rangle = \langle P\cdot Q\sqcap P\cdot Px,y\rangle$$

■

Multiplying (1) and (2) by P and Q we obtain equations relating $P\sqcap Q$ and $Q\sqcap P$, $P\sqcap Q$ and PQP, $Q\sqcap P$ and QPQ:

3.5. *Corollary.*

(7) $P\cdot Q\sqcap P\cdot P = P\sqcap Q\cdot QP$

(8) $P\cdot Q\sqcap P\cdot P = PQ\cdot P\sqcap Q$, and with Corollary 3.4, (5):

(9) $P\sqcap Q\cdot QP = PQ\cdot P\sqcap Q = PQP$;

(10) $QP\cdot Q\sqcap P = Q\cdot P\sqcap Q\cdot Q$

(11) $Q\sqcap P\cdot PQ = Q\cdot P\sqcap Q\cdot Q$, and with Corollary 3.4, (6):

(12) $QP\cdot Q\sqcap P = Q\sqcap P\cdot PQ$.

In the remainder of this section we want to present a simple condition under which the law of associativity

$$(P\sqcap Q)\sqcap R = P\sqcap(Q\sqcap R)$$

is valid. Trivially, this is the case if P, Q, R are pairwise commuting. There is a weaker condition, however; write

$$(P\sqcap Q)\sqcap R = E_0^{\perp}(R\cdot P\sqcap Q)$$

$$P\sqcap(Q\sqcap R) = E_0^{\perp}(Q\sqcap R\cdot P)$$

3.6. *Theorem.* For three projections P, Q, R in H with

$$PQ = QP \quad \text{and} \quad QR = RQ$$

the equality

$$(P\sqcap Q)\sqcap R = P\sqcap(Q\sqcap R)$$

which is the same as

$$(PQ)\sqcap R = P\sqcap(QR)$$

holds.

Proof. From $PQ = QP$, and Theorem 3.1, follows $R\cdot P\sqcap Q = R\cdot PQ = RQP$; from $QR = RQ$, and Theorem 3.1, follows $Q\sqcap R\cdot P = QR\cdot P = RQP$.

■

Remark. In Piron (1964, 1976), Piron studies a map which (in our notation) is given by

$$\Phi_P(Q) = P \sqcap Q$$

and he finds for instance that

$$\text{(i)} \quad \Phi_P \Phi_Q = \Phi_{P \wedge Q} \quad \text{iff } PQ = QP$$

$$\text{(ii)} \quad \Phi_P \Phi_Q = \Phi_Q \Phi_P \quad \text{iff } PQ = QP$$

These results resemble our criteria (4) and (6) of Theorem 3.1 above. If we evaluate (ii) at projections R, we see that $PQ = QP$ is equivalent to

$$P \sqcap (Q \sqcap R) = Q \sqcap (P \sqcap R) \qquad \text{for all } R$$

4. SEQUENTIAL EVENTS AND EFFECTIVE QL

It is interesting that although $\sqcap$ and $\sqcup$ do not share many of the algebraic properties $\wedge$ and $\vee$ have, they obey almost completely Mittelstaedt's (1979) effective quantum logic Q_{eff}, if we replace $\wedge$ and $\vee$ in Q_{eff} formulas by $\sqcap$ and $\sqcup$. In the following, we only have to reconsider rules of Q_{eff} into which $\sqcap$ and $\sqcup$ enter. The symbol „ denotes a meta-and.

Q_{eff}

(1.1) $P \leqslant P$

(1.2) $P \leqslant Q$„ $Q \leqslant R \Rightarrow P \leqslant R$

(2.1) $P \sqcap Q \leqslant P$

(2.2) $P \sqcap Q \leqslant Q$

is valid iff $PQ = QP$, see Theorem 3.1 (10).

(2.3) $R \leqslant P$„ $R \leqslant Q \Rightarrow R \leqslant P \sqcap Q$

is true, since $R \leqslant Q \leqslant P^{\perp} \vee Q$.

(3.1) $P \leqslant P \sqcup Q$

(3.2) $Q \leqslant P \sqcup Q$

is valid iff $PQ = QP$, from Theorem 3.1 (9) [cf. also (2.2) above].

(3.3) $P \leqslant R$„ $Q \leqslant R \Rightarrow P \sqcup Q \leqslant R$

is true, since $P^{\perp} \wedge Q \leqslant Q \leqslant R$ [cf. also (2.3) above].

(4.1) $(P \sqcap (P \to Q)) \leqslant Q$ "sequential modus ponens"

is true: $P \sqcap (P \to Q) = P \wedge (P^{\perp} \vee [P \to Q]) = P \wedge (P^{\perp} \vee [P^{\perp} \vee (P \wedge Q)]) = P \wedge (P^{\perp} \vee [P \wedge Q]) = P \wedge (P \to Q)$, so that there is no difference from the Q_{eff} modus ponens.

(4.2) $P \sqcap R \leqslant Q = P \to R \leqslant P \to Q$

is fulfilled because of $P \wedge R \leqslant P \sqcap R$, i.e., the antecedent here is even more restrictive than in Q_{eff}.

(4.3) $P \leqslant Q \to P \Rightarrow P \leqslant P \to Q$

states the symmetry of physical commensurability or, mathematically speaking, the symmetry of commutativity which, of course, is trivial in our

Hilbert space setting. [cf. also Theorem 3.1 (3)]. The following rule states that commensurability is closed with respect to $\sqcap$, $\sqcup$ and $\to$:

(4.4) $Q \leqslant P \to Q,, R \leqslant P \to R \Rightarrow Q * R \leqslant P \to (Q * R)$ with $* \in \{\sqcap, \sqcup, \to\}$

This will follow from (5.3) below. In the next three rules, 0 stands for the null space, representing falseness (and the full Hilbert space H represents truth):

(5.0) $0 \leqslant P$

(5.1) $P \sqcap P^{\perp} \leqslant 0$

(actually, of course, equality holds);

(5.2) $P \sqcap Q \leqslant 0 = P \to Q \leqslant P^{\perp}$

is true, since $P \wedge Q \leqslant P \sqcap Q$ [cf. (4.2) above].

(5.3) $P \leqslant Q \to P \Rightarrow P^{\perp} \leqslant Q \to P^{\perp}$

which says that $PQ = QP$ implies $P^{\perp}Q = QP^{\perp}$, i.e., commutativity is closed with respect to orthocomplementation. To finish the proof (4.4), note that (4.4) is fulfilled for $* = \to$ (see Q_{eff}); but then the identities $Q \sqcup R = Q^{\perp} \to R, Q \sqcap R = (Q^{\perp} \sqcap R^{\perp})^{\perp}$ together with (5.3) yield the desired result.

We know from Theorem 3.1 that Q_{eff} (2.2) and (3.2) are the only rules not generally valid for sequential events: they are equivalent to commutativity. Hence, in order to obtain a more general propositional calculus than Boolean logic, (2.2) or (3.2) must not hold—but then it is exactly this commutativity that should be sacrificed in an appropriate quantum logical propositional calculus!

Up to this point our discussion has shown where the introduction of $\sqcap$ and $\sqcup$ instead of $\wedge$ and $\vee$ destroys certain formal properties shared by, e.g., lattices. This is certainly a deficit of sequential algebras, and it is time to make up for this obvious lack in our syntax.

The advantages of the sequential calculus are as follows:

(a) $\sqcap$ and $\sqcup$ have a suggestive interpretation as "and then" and "or then," intuitively modeling a succession in time (see Section 2);

(b) $\sqcap$ is a noncommutative connective, which matches quantum theory well (see Sections 3 and 4); and, moreover,

(c) we regain a special but crucial case of (sub-) distributivity which permits the objectivation of events like "P and then Q" in the sense of Mittelstaedt (1976).

This means for instance for Young's two-slit-experiment that we may state objectively, i.e., *before* a measurement has been carried out:

> if a photon is observed at the spot x on the screen behind the two slits I and II, it can be inferred that it has passed before either *through I and then hit the screen at x* or *through II and then hit the screen at x.*

This language is justified by the following.

4.1. *Theorem* . For all projections P, Q in H

$$P \leqslant Q \sqcap P + Q^{\perp} \sqcap P,$$

more specifically:

$$P = Q \sqcap P + Q^{\perp} \sqcap P - I(P, Q)$$

where

$$I(P, Q) = P^{\perp} \sqcap (Q \sqcap P) = P^{\perp} \sqcap (Q^{\perp} P) = I(P, Q^{\perp})$$

Proof. The two projections on the right-hand side are orthogonal since $Q \sqcap P \leqslant Q$ and $Q^{\perp} \sqcap P \leqslant Q^{\perp}$, so that their sum (which may be written with $\vee$ or $\sqcup$ instead of $+$) is again a projection. From Definition 2.1 we see

$$\begin{aligned} Q \sqcap P + Q^{\perp} \sqcap P &= \{x | PQx = 0\}^{\perp} + \{x | PQ^{\perp} x = 0\}^{\perp} \\ &= \{x | PQx = 0 \quad \text{and} \quad PQ^{\perp} x = 0\}^{\perp} \\ &= \{x | PQx = 0 \quad \text{and} \quad Px = 0\}^{\perp} \\ &= (Q^{\perp} \sqcup P^{\perp} \wedge P^{\perp})^{\perp} = P \vee Q \sqcap P \\ &= P + P^{\perp} \sqcap (Q \sqcap P). \end{aligned}$$

$I(P, Q) = I(P, Q^{\perp})$ is easily seen. ■

There is the following suggestive reading of the implication in (4.1):

> every event P implies either "Q and then P" or "not-Q and then P."

If $P = P_x$ represents the event "photon hits x" and Q is the event "photon passes slit I," this reading rephrases the above objectivity of events occurring in Young's two-slit experiment.

5. OBJECTIVITY AND PROBABILITY OF SEQUENTIAL EVENTS

In this last section we shall make some remarks on the probability theory for sequential algebras, and on objectivity. For simplicity, we assume the physical system under consideration to be in a pure state represented by a unit vector φ. Then the probability of a property P is given by

$$w_\varphi(P)=\langle\varphi, P\varphi\rangle$$

In order for w_φ to be an appropriate probability function on a sequential algebra A, it has to fulfill five axioms (cf. Jauch, 1968, p.94; also Piron, 1964, 1976).

(W1) For all $P\in A$
$0\leqslant w_\varphi(P)\leqslant 1$

(W2) $w_\varphi(0)=0, w_\varphi(H)=1.$

(W3) For $P_1\leqslant P_2^\perp, P_1, P_2\in A$
$w_\varphi(P_1\sqcup P_2)=w_\varphi(P_1)+w_\varphi(P_2)$

(W4) For P_1, $P_2\in A$ with $w_\varphi(P_1)=w_\varphi(P_2)=1$
follows $w_\varphi(P_1\sqcap P_2)=1$

(W5) For $P>0$ follows $w_\varphi(P)>0$, and
if $P\neq Q$, then $w_\varphi(P)\neq w_\varphi(Q)$

(W1), (W2), and (W5) do not involve $\sqcap$ or $\sqcup$ and are therefore valid as in Mittelstaedt's probability theory for Q_{eff}. As for (W3), observe that from $P_1\leqslant P_2$ follows

$$P_1\sqcup P_2=P_1\vee(P_1^\perp\wedge P_2)=P_1\vee P_2=P_1+P_2$$

and (W4) is obvious from $P_1\sqcap P_2\geqslant P_1\wedge P_2$. Applying (W3) for $P_1=Q\sqcap P, P_2=Q^\perp\sqcap P$, Theorem 4.1 gives

(I) $w_\varphi(P)=w_\varphi(Q\sqcap P)+w_\varphi(Q^\perp\sqcap P)-w_\varphi(I(P,Q))$

with $I(P,Q)=P^\perp\sqcap(Q\sqcap P)$. Here $w_\varphi(Q\sqcap P)$ is the probability that the compound event "Q and then P" will happen—even before any measurement is made.

This formulation and formula (I) are not in contradiction to the physical fact of interference. To see this, let us write down the usual formulas containing interference terms [for the following see also Mittelstaedt's (1976) discussion].

Let

$$w_\varphi(P,Q):=\langle\varphi, QPQ\varphi\rangle \tag{5.1}$$

be the probability for our system to have the property P *after the property Q has been stated*, if previously the system had been in the state φ. This latter interpretation is especially transparent when we write

$$w_\varphi(P,Q)=\langle\varphi_Q, P\varphi_Q\rangle \qquad \text{with} \quad \varphi_Q=Q\varphi \tag{5.2}$$

meaning that a measurement or statement of Q changes the state φ into φ_Q (unnormalized), and after this measurement there is a "conditional" probability for P. This is, *cum grano salis*, the von Neumann–Lüders interpretation (von Neumann, 1955; Lüders, 1951; see also Bub's discussion, 1977). This interpretation takes QPQ, properly normalized by the trace (QPQ), as the conditional probability operator for "the probability $w_Q(P)$ of P, given Q."

In contrast, but not in contradiction, to (5.1) and (5.2)

$$w_\varphi(Q \sqcap P) = \langle \varphi, Q \sqcap P \varphi \rangle$$

is the probability for the compound sequential event "P after Q" or "Q and then P," if previously the system has been in the state φ. This is regardless of whether Q has been stated or not; and consequently we should expect

$$w_\varphi(Q \sqcap P) \geqslant w_\varphi(P, Q) \tag{5.3}$$

This is in fact true, for $Q \sqcap P$ "reduces" the Hermitian operator QPQ:

$$QPQ = Q \sqcap P \cdot QPQ = QPQ \cdot Q \sqcap P \tag{5.4}$$

(cf. also the end of Section 2). Equation (5.4) is an equality from the spectral analysis of operators [see Sz. -Nagy, 1967, formula (3) on page 23].

Using our result (6) of Corollary 3.4:

$$QPQ = Q \cdot P \sqcap Q \cdot Q$$

we see also that

$$w_Q(P) = w_Q(P \sqcap Q) \tag{5.5}$$

i.e., the probabilities of P and of $P \sqcap Q$ given Q, are the same.

The probabilities of (5.1) determine the probabilities $w_\varphi^{\text{int}}(P, Q)$ of interference via formula

(II) $w_\varphi(P) = w_\varphi(P, Q) + w_\varphi(P, Q^\perp) + w_\varphi^{\text{int}}(P, Q)$

where

$$w_\varphi^{\text{int}}(P, Q) = \langle \varphi, (QPQ^\perp + Q^\perp PQ)\varphi \rangle \tag{5.6}$$

can be $\geqslant 0$ or $\leqslant 0$.

In addition to $w_\varphi(P, Q)$ and $w_\varphi(Q \sqcap P)$, some authors give a probability $w_\varphi(Q \wedge P)$ by means of

$$w_\varphi(Q \wedge P) = \langle \varphi, Q \wedge P \varphi \rangle \tag{5.7}$$

also to the "event" $Q \wedge P = P \wedge Q$. This is then interpreted as a probability that the system has the property P, if it had before the property Q and this property Q is maintained even after the measurement of the property P.

From $Q \wedge P \leqslant QPQ$ we see that

$$w_\varphi(Q \wedge P) \leqslant w_\varphi(P,Q)$$

and so always

(III) $w_\varphi(P) \geqslant w_\varphi(Q \wedge P) + w_\varphi(Q^\perp \wedge P) + w_\varphi^{\text{int}}(P,Q)$

To sum up, we have the following situation: in general, the double inequality

$$w_\varphi(Q \wedge P) \leqslant w_\varphi(P,Q) \leqslant w_\varphi(Q \sqcap P)$$

holds, and if there is *one* equality sign here for all φ, the other inequality becomes an equality, too (Theorem 3.1). In this case, P and Q commute, and $w_\varphi^{\text{int}}(P,Q) = 0$. Now, in more detail, we have the following.

(i) If we want to objectivize $Q \wedge P$, we must have (cf. Mittelstaedt, 1976 p. 159)

$$P \leqslant Q \wedge P + Q^\perp \wedge P$$

which contradicts III, i.e., experimental quantum theory. Therefore,

(i.1) do not objectivize (Copenhagen view);

or

(i.2) change and/or restrict the logic (QL);

or

(i.3) write and interpret $Q \wedge P$ as a sequential event $Q \wedge P = Q \sqcap (Q^\perp \sqcup P)$ and replace $P \leqslant Q \wedge P + Q^\perp \wedge P$, which is invalid, by $P \leqslant Q \sqcap P + Q^\perp \sqcap P$, which is true (Theorem 4.1).

(ii) Equality (II), which is in accordance with quantum theory, may be read in two ways:

(ii.1) *Copenhagen interpretation.* $w_\varphi(P)$ is the probability of finding the property P *after* the measuring process, when the transition from the initial state to an eigenstate of P has taken place. This probability is at most equal to the sum of two probabilities $w_\varphi(P,Q)$ and $w_\varphi(P,Q^\perp)$ which, in their interpretation, involve the notion of a *prior statement of* Q before a (conditional) probability can be assigned to P. The remainder $w_\varphi^{\text{int}}(P,Q)$, the "interference probability," may be $\leqslant 0$!

(ii.2) *Mittelstaedt's interpretation.* All possible properties P of a system even in QT are simultaneously objective prior to measurement, and $w_\varphi(P)$ is the "objective" probability before the measurement has been carried out.

(ii.3) *Comparison of (ii.1) and (ii.2).* The price to pay for the Copenhagen interpretation is a counterintuitive lack of objectivity, and, more important, the willingness to accept the rather restrictive and ambivalent Copenhagen dogma which claims, *in nuce*, that although the classical logic remains true, certain rules and theorems (distributivity) are not applicable in QT. For Mittelstaedt, in order to avoid difficulties with experimental evidence, the probability calculus has to be based on a "weaker" logic than Boolean logic, which, for instance, does not admit general distributivity. This approach has been criticized from a metalogical standpoint (Hübner, 1964) claiming that logic must not depend on experience. Since there exists the exactly opposite view, too, we consider this controversy a draw. Although Mittelstaedt's Q_{eff} leads to a quantum probability theory quite naturally, the right-hand sides of (II) or (III) above have a flavor retained from the Copenhagen view: they still involve a prior stating or measuring of Q before property P "has" a probability, so that the right-hand terms individually are not fully objectivized. Besides, QPQ in (II) is in general not a property (projection) but only an observable (Hermitian operator), so that (ii.2) seems to contain a tinge of inconsequence or, at least, a lack of symmetry. Moreover, since $QPQ^{\perp} + Q^{\perp}PQ$ is not always a positive observable, $w_{\varphi}^{\text{int}}(P, Q)$ cannot be intepreted as a probability.

(iii) The sequential calculus of projections proposed in this paper, besides incorporating the essential feature of QT, incommensurability, contains the important identity

$$P = Q \sqcap P + Q^{\perp} \sqcap P - I(P, Q)$$

where P, $Q \sqcap P$, $Q^{\perp} \sqcap P$, and $I(P, Q)$ all represent properties (i.e., are projections) of the system, and they each have a probability before any measurement is made.

We conclude this paper on a note of self-criticism. The logic of sequential events as sketched above relies upon the particle picture of quantum mechanics on the one hand: only then do Q and $Q^{\perp}$ both have objective meaning. On the other hand, it takes into account the wave picture by introducing an interference term in Theorem 4.1. The occurrence of interference, however, contradicts the particle picture; it contradicts, in other words, the initial building stone of our theory. Even if it is conceded that this kind of double-talk reflects an allegedly intrinsic particle–wave duality of quantum phenomena, a weakening of Boolean logic as done in different ways by Birkhoff and von Neumann, Reichenbach, and Mittelstaedt, has a rather more serious methodological flaw: the empirical content is weakened, too, since weaker laws decrease the possibilities of testing and falsifying a theory (this is Feyerabend's criticism).

In the present approach, by giving priority to sequential connectives over the logical connectives, we inserted a syntax for "succession" at the bottom of the theory, and derived some of its consequences leading to a probability semantics. Logic proper thus becomes secondary to our trying to capture some essential physical features of quantum theory: incommensurability, an objectivistic probability structure, and interference.

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